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A PECULIAR SET OF PROBLEMS IN LINEAR STRUCTURAL MECHANICS

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Abstract—Peculiar analytical results encountered in the literature on continuously supported structures are pointed out and investigated. They are: (1) for a finite beam which rests on a Winkler foundation and is centrally loaded by a concentrated force P, the points of separation of beam and base are not affected by the magnitude of the load, and (2) according to the well known solution for an infinite (or a semi-infinite) beam attached to a Winkler foundation and subjected to a concentrated load P, the location of the zero points for deflections and bending moments do not depend on P. Intuitively, it is expected that these entities should depend on the load P. At first, it is shown that these peculiar results are a consequence of the linearity of the respective formulations and that the same feature will also be exhibited for other linear foundation models (for example, the Pasternak model or the elastic continuum). To clarify these analytical features, the above problems are re-analysed by including a non-linearity in the Winkler foundation response. To simplify the analyses, a bi-linear response is used. It was found that: (1) for the finite beam that rests on the base, the intensity of the load does affect the location of the point of separation of beam and base; namely, that an increasing load and a "stiffening" base decrease the region of contact; (2) for the infinite beam that is attached to the base the situation is similar; an increasing load and a "stiffening" of the base decrease the distances of the zero locations and reduce the maximum bending moment, whereas for a "softening" base these distances increase as compared to the linear case.

INTRODUCTION AND STATEMENT OF PROBLEM

According to the analyses presented by Hayashi (1921, p. 64) and Hetényi (1946, p. 54) for a finite beam which rests on a linear Winkler foundation and is subjected to a force P, as shown in Fig. 1, the length of contact between beam and base is:

$$l = \frac{\pi}{\beta}. \tag{1}$$

where

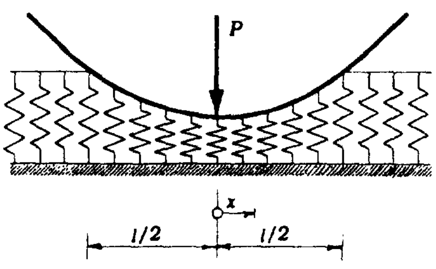

Fig. 1. Beam on Winkler base with lift-off.

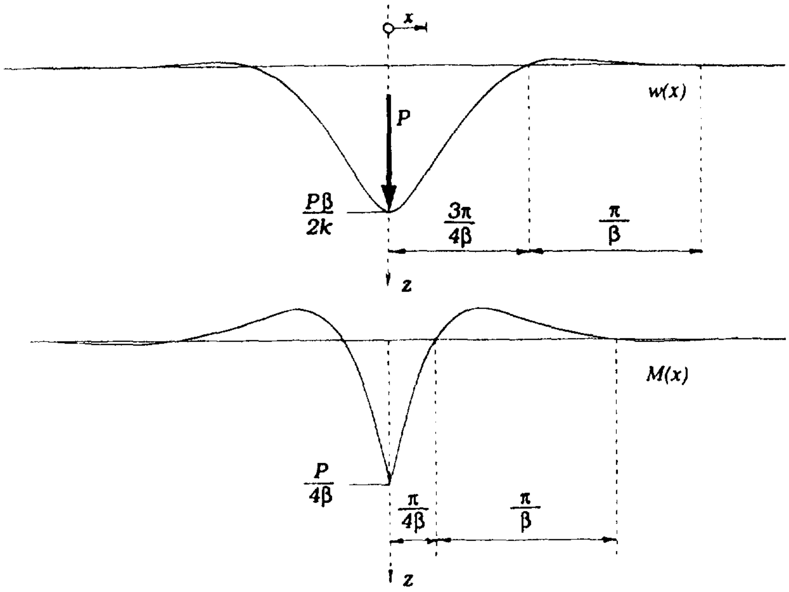

Fig. 2. Infinite beam attached to a Winkler base.

$$\beta = \sqrt[4]{\frac{k}{4EI}}. \tag{2}$$

k is the foundation modulus and EI is the bending stiffness of the beam. Thus, the length of contact is the same for any, small or large, value of the load P, a rather peculiar result.

A similar feature is observed when analysing an infinite or semi-infinite beam attached to a linear Winkler foundation (Hetényi, 1946, Chapter II). Namely, it is found that the locations of the points where the beam deflection (or bending moment, or shearing force) are zero do not depend on P. As an example, the results for the infinite beam subjected to a load P are shown in Fig. 2. Note that the first location where the deflection becomes zero is at

$$l = \frac{3\pi}{4\beta}. \tag{3}$$

Intuitively, one would expect that the length of contact for the problem in Fig. 1, or the location of the zero points in Fig. 2, should depend on the load P.

The explanation of these peculiar results is based on the fact that the analytical formulation for each of these examples is a linear boundary value problem with the load P as a non-homogeneity. Thus, the resulting deflection expression is of the form

$$w(x) = P \cdot f(x, \beta). \tag{4}$$

Therefore, the locations $\bar{x}$ where $w = 0$, for $P \neq 0$, is determined from the condition

$$f(\bar{x}, \beta) = 0, \tag{5}$$

that does not contain P.

The same argument applies to bending moments and shearing forces, which are obtained as derivatives of $w(x)$. Similar results will also be obtained for different linear foundation models, like those of Pasternak and the elastic continuum. For examples refer to Biot (1937), Reissner (1937), Kerr and Coffin (1991), and the books by Korenev (1954) and Selvadurai (1979).

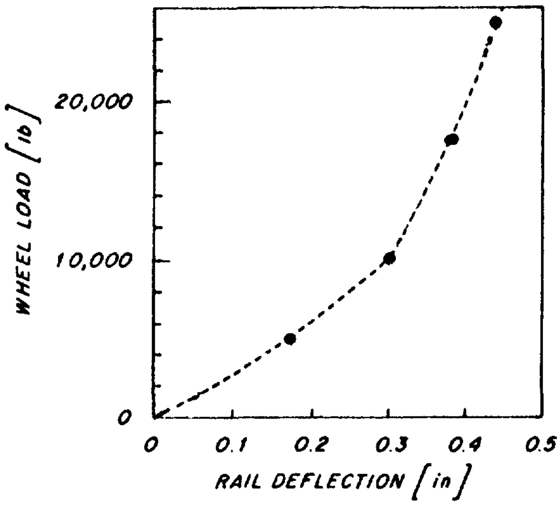

Fig. 3. Test results (Talbot, 1919).

In order to study the sensitivity of the results, as given in eqns (1) and (3), a non-linearity has to be introduced.

In actual engineering problems the base often stiffens with an increasing load. As an example, the response of a railroad track when subjected to a wheel load P, obtained from tests, is shown in Fig. 3. Note that the recorded load vs displacement relation is non-linear. It is of interest to determine what effect this non-linearity has on the analytical results discussed above, and to establish the accuracy of the analyses that are based on the linear Winkler foundation. This is discussed in the following.

ANALYTICAL PRELIMINARIES

The differential equation for an elastic beam of constant cross-section is

$$EIw^{iv} + p(x) = q(x), \tag{6}$$

where $w(x)$ is the deflection at x, $q(x)$ is a prescribed distributed load and $p(x)$ is the contact pressure between the beam and the base.

In 1867 Winkler introduced the linear relation

$$p(x) = kw(x). \tag{7}$$

The results discussed in the Introduction are based on this assumption. Only for small loads P is eqn (7) suitable for representing the test data in Fig. 3. For a larger range of loads, a better representation is (Kerr, 1969)

$$p(x) = \frac{k}{n}\sinh\,[nw(x)], \tag{8}$$

where k and n are the foundation parameters. Note that for sufficiently small values of w, the above relation reduces to eqn (7). Utilizing eqn (8), eqn (6) becomes

$$EIw^{iv} + \frac{k}{n}\sinh\,[nw(x)] = q, \tag{9}$$

a non-linear ordinary differential equation. No closed-form general solution is available for this equation.

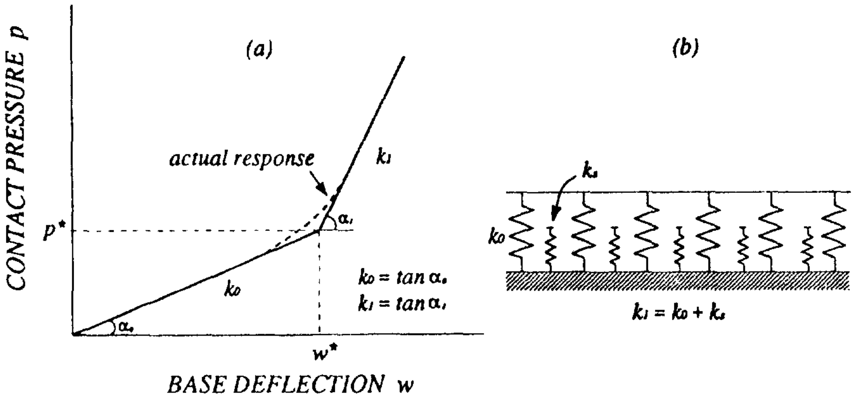

Fig. 4. Bi-linear approximation of base response and its mechanical model.

To simplify the analyses for the problems under consideration, a bi-linear representation of the non-linear pressure–deflection relationship is used, as shown in Fig. 4(a), where k_0 and k_1 are the base parameters for the "initial" and "stiffened" zones, respectively. This bi-linear response is expressed analytically as

$$p(x) = k_0 w(x) \qquad \text{for} \quad w \leqslant w^* \tag{10a}$$

$$p(x) = k_0 w^* + k_1 [w(x) - w^*] \quad \text{for} \quad w \geqslant w^*, \tag{10b}$$

where w^* is the deflection beyond which k_0 becomes k_1. A mechanical model for this bi-linear response is shown in Fig. 4(b). Since the base response is represented by two independent spring layers that respond linearly, it follows that $k_1 = k_0 + k_s$.

FINITE BEAM RESTING ON A BI-LINEAR BASE

At first, note that for small P the base responds like a linear Winkler foundation with $k = k_0$. Utilizing symmetry, the origin of the coordinate x is placed at the load P, as shown in Fig. 1. The governing differential equation for the beam is

$$EIw^{iv} + k_0 w = 0 \quad 0 \leqslant x \leqslant l/2. \tag{11}$$

The four integration constants and the unknown distance to the point of lift-off, $l/2$, are determined from the five boundary conditions

$$w'(0) = 0; \qquad w'''(0) = \frac{P}{2EI}$$

$$w''(l/2) = 0; \quad w'''(l/2) = 0; \quad w(l/2) = 0. \tag{12}$$

Utilizing the first four conditions, the deflection expression, as given by Hetényi (1946, p. 53), is

$$w(x) = \frac{P\beta_0}{2k_0} \frac{1}{\sinh(\beta_0 l) + \sin(\beta_0 l)} \{\cosh(\beta_0 x)\cos[\beta_0(l-x)] + \cos(\beta_0 x)\cosh[\beta_0(l-x)]$$

$$-\sinh(\beta_0 x)\sin[\beta_0(l-x)] + \sin(\beta_0 x)\sinh[\beta_0(l-x)] + 2\cosh(\beta_0 x)\cos(\beta_0 x)\}, \tag{13}$$

where $\beta_0 = \sqrt[4]{k_0/(4EI)}$. The as yet unknown distance to the point where the beam separates from the base is determined using the fifth condition of eqns (12). The resulting equation is satisfied when $\beta_0 l = \pi$, or

$$\frac{l}{2} = \frac{\pi}{2\beta_0}. \tag{14}$$

The maximum deflection takes place under the load at $x = 0$. Since there the deflection of the beam and the base are identical, the range of P for which the linear analysis is valid is determined from

$$w(0) \leqslant w^*. \tag{15}$$

It then follows from eqn (13) that the linear analysis is valid when

$$P \leqslant P^* = \frac{2k_0 w^*}{\beta_0} \frac{\sinh(\beta_0 l) + \sin(\beta_0 l)}{\cosh(\beta_0 l) + \cos(\beta_0 l) + 2}. \tag{16}$$

Next consider the case where $w(x) > w^*$. This case corresponds to the situation when $P > P^*$ and thus the deflections in the vicinity of P are in the non-linear range. Using the bi-linear approximation for the base response in the vicinity of the load, the beam that initially deflected linearly with a foundation modulus k_0 reaches the transition deflection w^*, beyond which the foundation modulus changes to $k_1 > k_0$.

To simplify the analysis, symmetry is utilized. Then the beam on the right-hand-side is divided into three domains. The outer region, not in contact with the base, will be stress-free and is not included in the analysis. Each of the two regions on the right side in contact with the base is assigned its own coordinates x and ξ, as shown in Fig. 5.

The two governing differential equations are

$$\begin{aligned} EIw_x^{iv} + k_1 w_x &= (k_1 - k_0)w^* \qquad 0 \leqslant x \leqslant a \\ EIw_\xi^{iv} + k_0 w_\xi &= 0 \qquad 0 \leqslant \xi \leqslant b, \end{aligned} \tag{17}$$

where $w_x = w(x)$ and $w_\xi = w(\xi)$. For the determination of the eight integration constants and the two as yet unknown lengths (a, b), 10 boundary and matching conditions are needed. They are:

$$\begin{array}{llll} w_x'(0) = 0 & \text{(a)}, & w_x'''(0) = P/(2EI) & \text{(b)}, \\ w_x(a) = w_\xi(0) & \text{(c)}, & w_\xi(0) = w^* & \text{(d)}, \\ w_x'(a) = w_\xi'(0) & \text{(e)}, & w_x''(a) = w_\xi''(0) & \text{(f)}, \\ w_x'''(a) = w_\xi'''(0) & \text{(g)}, & w_\xi(b) = 0 & \text{(h)}, \\ w_\xi''(b) = 0 & \text{(i)}, & w_\xi'''(b) = 0 & \text{(j)}. \end{array} \tag{18}$$

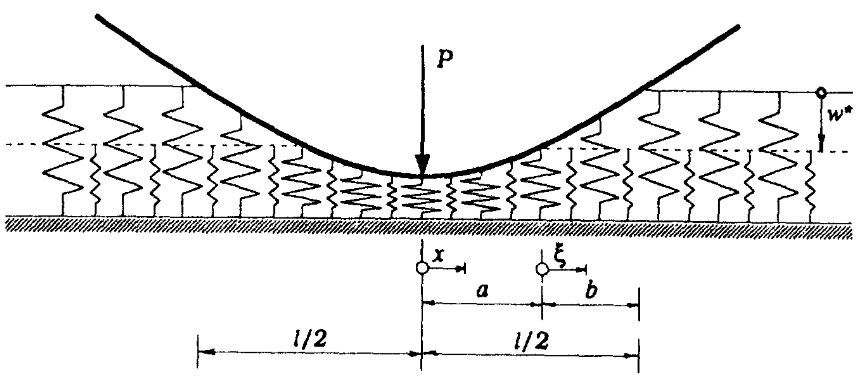

Fig. 5. Finite beam on bi-linear base.

The general solutions of the governing differential equations are:

$$w(x) = [(k_1-k_0)/k_1]w^*+A_1\cos(\beta_1 x)\cosh(\beta_1 x)+A_2\sin(\beta_1 x)\cosh(\beta_1 x) \\ +A_3\cos(\beta_1 x)\sinh(\beta_1 x)+A_4\sin(\beta_1 x)\sinh(\beta_1 x)$$

$$w(\xi) = A_5\cos(\beta_0\xi)\cosh(\beta_0\xi)+A_6\sin(\beta_0\xi)\cosh(\beta_0\xi) \\ +A_7\cos(\beta_0\xi)\sinh(\beta_0\xi)+A_8\sin(\beta_0\xi)\sinh(\beta_0\xi), \quad (19)$$

where

$$\beta_0 = \sqrt[4]{k_0/(4EI)} \quad \text{and} \quad \beta_1 = \sqrt[4]{k_1/(4EI)}. \quad (20)$$

Substituting the above expressions into the boundary and matching conditions, except for eqns (e) and (g), yields the eight integration constants:

$$A_1 = \frac{P\Delta_1\beta_1^3(\bar{c}\bar{s}-cs)+2w^*(\Delta_1\beta_1^2k_0c\bar{c}-\Delta_2\beta_0^2k_1s\bar{s})}{2\Delta_1\beta_1^2k_1(\bar{s}^2+c^2)}, \quad A_2 = -A_3 = \frac{P\beta_1}{2k_1},$$

$$A_4 = \frac{-P\Delta_1\beta_1^3(\bar{c}\bar{s}+cs)+2w^*(\Delta_1\beta_1^2k_0s\bar{s}+\Delta_2\beta_0^2k_1c\bar{c})}{2\Delta_1\beta_1^2k_1(\bar{s}^2+c^2)}, \quad A_5 = w^*,$$

$$A_6 = +\frac{w^*}{\Delta_1}[\sin(\beta_0 b)\sinh(\beta_0 b)-\cos(\beta_0 b)\cosh(\beta_0 b)],$$

$$A_7 = -\frac{w^*}{\Delta_1}[(\sin(\beta_0 b)\sinh(\beta_0 b)+\cos(\beta_0 b)\cosh(\beta_0 b)],$$

$$A_8 = +\frac{w^*}{\Delta_1}[\cos(\beta_0 b)\sinh(\beta_0 b)-\cosh(\beta_0 b)\sin(\beta_0 b)], \quad (21)$$

where

$$s = \sin(\beta_1 a) \quad \bar{s} = \sinh(\beta_1 a)$$
$$c = \cos(\beta_1 a) \quad \bar{c} = \cosh(\beta_1 a)$$
$$\Delta_1 = \cos(\beta_0 b)\sinh(\beta_0 b)+\sin(\beta_0 b)\cosh(\beta_0 b)$$
$$\Delta_2 = \cos(\beta_0 b)\sinh(\beta_0 b)-\sin(\beta_0 b)\cosh(\beta_0 b). \quad (21')$$

The unknown lengths (a, b) are determined from the two remaining matching conditions, eqns (18e, g). They yield two non-linear simultaneous algebraic equations for these two unknowns:

$$-(A_1-A_4)\sin(\beta_1 a)\cosh(\beta_1 a)+(A_1+A_4)\cos(\beta_1 a)\sinh(\beta_1 a) \\ +(A_2+A_3)\cos(\beta_1 a)\cosh(\beta_1 a)+(A_2-A_3)\sin(\beta_1 a)\sinh(\beta_1 a) = (A_6+A_7)\beta_0/\beta_1 \quad (22)$$

$$-(A_1+A_4)\sin(\beta_1 a)\cosh(\beta_1 a)-(A_2+A_3)\sin(\beta_1 a)\sinh(\beta_1 a) \\ +(A_4-A_1)\cos(\beta_1 a)\sinh(\beta_1 a)+(A_2-A_3)\cos(\beta_1 a)\cosh(\beta_1 a) = (A_6-A_7)\beta_0^3/\beta_1^3. \quad (23)$$

A computer program was devised for solving these two equations. They were solved, at first, for $\gamma = k_1/k_0 = 2$. The results, for a range of load parameters λ, are shown in Fig.

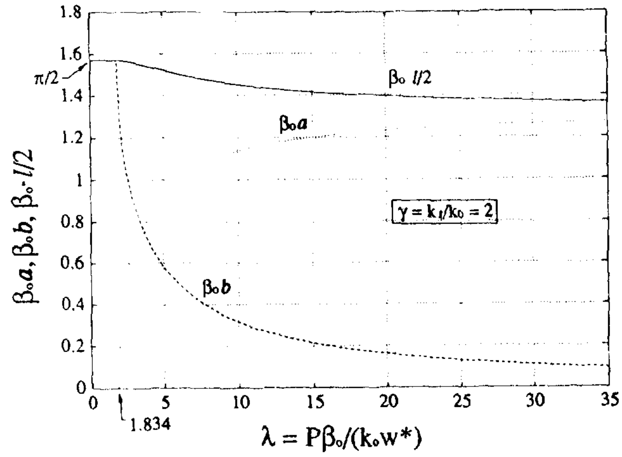

Fig. 6. Dependence of a, b and $l/2 = (a+b)$ on the Load Parameter λ.

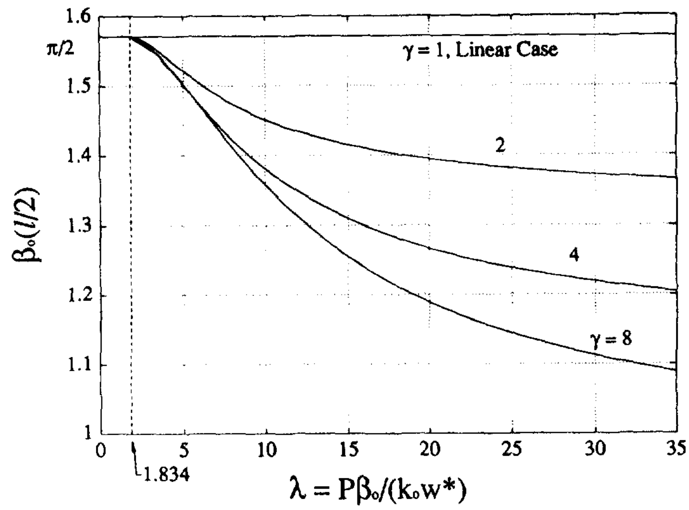

Fig. 7. Dependence of $l/2$ on λ and the bi-linear parameter $\gamma = k_1/k_0$.

6. Note that for $P < P^*$, i.e. for $\lambda < \lambda^* = P^*\beta_0/(k_0 w^*)$ the value of $\beta_0 l$ is equal to π, as predicted by eqn (14). According to eqn (16), the corresponding value, which is the transition point from the linear to the bi-linear response, is

$$\lambda^* = \frac{P^*\beta_0}{k_0 w^*} = 1.834. \tag{24}$$

The dependence of the contact length $l/2$ on the load parameter λ, for various values of the "non-linearity" parameters $\gamma = k_1/k_0$, are presented in Fig. 7. The case $\gamma = 1$, shown as a horizontal straight line, corresponds to the linear case. According to Fig. 7, for $\lambda \leqslant 1.834$, the half contact length $l/2 = \pi/(2\beta_0)$, the linear case. However, for $P > P^*$, when the beam enters the non-linear response range with the foundation modulus k_1, the curves separate. Note that with increasing k_1/k_0, i.e. with increasing stiffening of the base response, and increasing load P, the contact region l decreases.

To demonstrate the type of non-linearity represented by the chosen values $\gamma = k_1/k_0$, the corresponding base pressure vs deflection graphs are shown in Fig. 8.

INFINITE BEAM ATTACHED TO A BI-LINEAR BASE

This case is shown in Fig. 2. As in the previous section for small P, the base responds like a linear Winkler foundation with $k = k_0$. This is a well known case. Utilizing symmetry it is governed by the differential equation

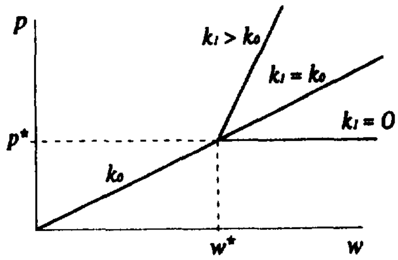

Fig. 8. Effect of k_1/k_0 on p vs w.

$$EIw^{iv} + k_0 w = 0 \quad 0 < x < \infty \tag{25}$$

and the corresponding boundary conditions

$$w'(0) = 0; \quad w'''(0) = \frac{P}{2EI}$$
$$\lim_{x \to \infty} \{w, w'\} \to \text{finite}. \tag{26}$$

The solution is

$$w(x) = \frac{P\beta_0}{2k_0} e^{-\beta_0 x}(\cos\beta_0 x + \sin\beta_0 x) \quad 0 \leqslant x \leqslant \infty. \tag{27}$$

To determine the locations where the deflections are zero, we set $w(\bar{x}) = 0$. This condition reduces to

$$\tan\beta_0\bar{x} = -1 \tag{28}$$

with the roots

$$\bar{x} = \frac{3\pi}{4\beta_0}, \quad \frac{7\pi}{4\beta_0}, \ldots \tag{28'}$$

The corresponding bending moments are

$$M(x) = \frac{P}{4\beta_0} e^{-\beta_0 x}\{\cos\beta_0 x - \sin\beta_0 x\} \quad 0 \leqslant x \leqslant \infty. \tag{29}$$

Setting $M(\bar{x}) = 0$ and solving for $\bar{x}$ yields

$$\bar{x} = \frac{\pi}{4\beta_0}, \quad \frac{5\pi}{4\beta_0}, \ldots \tag{30}$$

These results are presented in Fig. 2.

The maximum deflection takes place under the load at $x = 0$. The range of P for which the linear analysis is valid is obtained from the condition $w(0) \leqslant w^*$. It then follows from eqn (27) that the linear analysis is valid when

$$P \leqslant \frac{2k_0 w^*}{\beta_0} = P^*. \tag{31}$$

Next consider the case when $w(x) > w^*$. This corresponds to the situation when $P > P^*$, and thus the deflections in the vicinity of P are in the non-linear range. To simplify

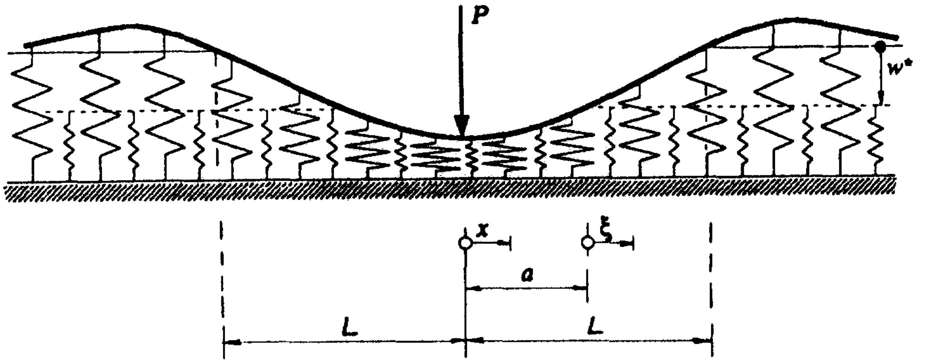

Fig. 9. Infinite beam on bi-linear base.

the analysis, symmetry is utilized, the beam is divided into two domains and each domain is assigned its own coordinates x and ξ, as shown in Fig. 9.

The two governing differential equations are

$$\begin{aligned} EIw_x^{\mathrm{iv}} + k_1 w_x &= (k_1 - k_0)w^* \qquad 0 \leqslant x \leqslant a \\ EIw_\xi^{\mathrm{iv}} + k_0 w_\xi &= 0 \qquad\qquad\qquad 0 \leqslant \xi \leqslant \infty, \end{aligned} \tag{32}$$

where $w_x = w(x)$ and $w_\xi = w(\xi)$. For the determination of the eight integration constants and the as yet unknown length, a, the following nine boundary and matching conditions are prescribed:

$$\begin{aligned} &w_x'(0) = 0, \qquad w_x'''(0) = \frac{P}{2EI}, \\ &w_x(a) = w_\xi(0), \quad w_\xi(0) = w^*, \quad w_x'(a) = w_\xi'(0), \\ &w_x''(a) = w_\xi''(0), \quad w_x'''(a) = w_\xi'''(0), \quad \lim_{\xi\to\infty} \{w_\xi, w_\xi'\} \to \text{finite}. \end{aligned} \tag{33}$$

This case was solved by Kerr and Shenton (1986). The general solution is

$$\begin{aligned} w(x) = [(k_1 - k_0)/k_1]w^* + A_1 \cos(\beta_1 x)\cosh(\beta_1 x) + A_2 \sin(\beta_1 x)\cosh(\beta_1 x) \\ + A_3 \cos(\beta_1 x)\sinh(\beta_1 x) + A_4 \sin(\beta_1 x)\sinh(\beta_1 x) \\ w(\xi) = e^{-\beta_0\xi}[A_5 \cos(\beta_0\xi) + A_6 \sin(\beta_0\xi)] + e^{\beta_0\xi}[A_7 \cos(\beta_0\xi) + A_8 \sin(\beta_0\xi)]. \end{aligned} \tag{34}$$

Substituting these expressions into eqns (33), except for $w_x''(a) = w_\xi''(0)$, yields the eight integration constants

$$\begin{aligned} A_1 &= \frac{1}{\Delta}\left\{\frac{P\beta_1}{2k_1}[\bar{s}^2 - s^2 + \kappa(\bar{s}\bar{c} - sc)] + \frac{w^*}{\kappa}\left[s\bar{s} + \frac{1}{\kappa^3}(s\bar{c} + \bar{s}c + \kappa c\bar{c})\right]\right\}, \\ A_2 &= \frac{P\beta_1}{2k_1}, \quad A_3 = \frac{-P\beta_1}{2k_1}, \\ A_4 &= \frac{1}{\Delta}\left\{\frac{-P\beta_1}{2k_1}[\bar{s}^2 + s^2 + \kappa(sc + \bar{s}\bar{c})] - \frac{w^*}{\kappa}\left[c\bar{c} + \frac{1}{\kappa^3}(\bar{s}c - s\bar{c} - \kappa s\bar{s})\right]\right\}, \\ A_5 &= w^*, \quad A_6 = \frac{1}{\Delta}\left\{\frac{-P\beta_1}{k_1}\kappa^2 s\bar{s} + w^*\kappa\left[\bar{s}^2 + c^2 + \frac{1}{\kappa^3}(\bar{s}\bar{c} - sc)\right]\right\}, \\ A_7 &= A_8 = 0. \end{aligned} \tag{35}$$

Table 1. Non-dimensional length $\beta_0 a$ for various λ and γ values

$\lambda = \frac{P\beta_0}{k_0 w^*}$	$\gamma = 0.01$ ($\kappa = 0.316$)	$\gamma = 0.25$ ($\kappa = 0.707$)	$\gamma = 0.5$ ($\kappa = 0.841$)	$\gamma = 2.0$ ($\kappa = 1.189$)	$\gamma = 4.0$ ($\kappa = 1.414$)	$\gamma = 8.0$ ($\kappa = 1.682$)
2	0	0	0	0	0	0
4	1.881	1.371	1.192	0.846	0.695	0.565
6	3.106	1.850	1.565	1.073	0.874	0.706
8	3.981	2.129	1.782	1.208	0.981	0.792
10	4.601	2.317	1.929	1.299	1.055	0.851
15	5.563	2.607	2.156	1.443	1.170	0.945
20	6.127	2.779	2.291	1.578	1.239	1.001
25	6.508	2.894	2.382	1.585	1.285	1.039
30	6.788	2.979	2.448	1.627	1.319	1.067
35	7.004	3.044	2.498	1.659	1.345	1.088
40	7.177	3.095	2.538	1.684	1.365	1.105

where

$$s = \sin(\beta_1 a), \qquad \bar{s} = \sinh(\beta_1 a),$$
$$c = \cos(\beta_1 a), \qquad \bar{c} = \cosh(\beta_1 a),$$
$$\Delta = sc + \bar{s}\bar{c} + \kappa(\bar{s}^2 + c^2), \quad \kappa = \beta_1/\beta_0 = \sqrt[4]{k_1/k_0} = \sqrt[4]{\gamma},$$
$$\beta_0 = \sqrt[4]{k_0/(4EI)}, \qquad \beta_1 = \sqrt[4]{k_1/(4EI)}. \tag{35'}$$

The unknown length a is determined from the remaining condition $w_x''(a) = w_\xi''(0)$. The resulting equation is, for $\kappa \neq 0$,

$$\lambda\kappa(s+\kappa c)(\bar{s}+\kappa\bar{c}) - [\bar{s}\bar{c}(\kappa^2+1)^2 - sc(\kappa^2-1)^2 + 2\bar{s}^2\kappa(\kappa^2+1) - 2s^2\kappa(\kappa^2-1) + 2\kappa^3] = 0. \tag{36}$$

It is a highly non-linear algebraic equation for the one variable $\beta_1 a$ or $\kappa\beta_0 a$, which depends only on the non-dimensional parameter

$$\lambda = \frac{P\beta_0}{k_0 w^*}. \tag{37}$$

Equation (36) was solved numerically for $\beta_0 a$ using the International Mathematics and Statistics Library (IMSL) subroutine ZSCNT for a wide range of λ and γ values. Some of the obtained results are shown in Table 1.

Next, the points at which the deflection is $= 0$ (length L) are determined, by plotting the resulting w_ξ. For this we utilize the a values from Table 1 in order to establish the origin of the ξ-coordinate for a given load parameter λ and $\gamma = k_1/k_0$. The calculated values $\beta_0 L$ are presented graphically in Fig. 10.

The linear case, $\gamma = 1$, is represented by a horizontal straight line for which $\beta_0 L = 3\pi/4$, in agreement with Fig. 2. Thus, as discussed previously, for this case L does not vary with P, small or large.

For $\gamma \neq 1$ (bi-linear base response) and small P values, L is independent of P. However, as P exceeds P^* and the base exhibits a bi-linear response, the curves for different $\gamma = k_1/k_0$ separate, as shown in Fig. 10. The λ value at which the separation takes place is determined from eqn (31). It is

$$\lambda^* = \frac{P^*\beta_0}{k_0 w^*} = 2. \tag{38}$$

The case $\gamma = 0$ takes place when $k_1 = 0$. It corresponds to a horizontal line for $w > w^*$

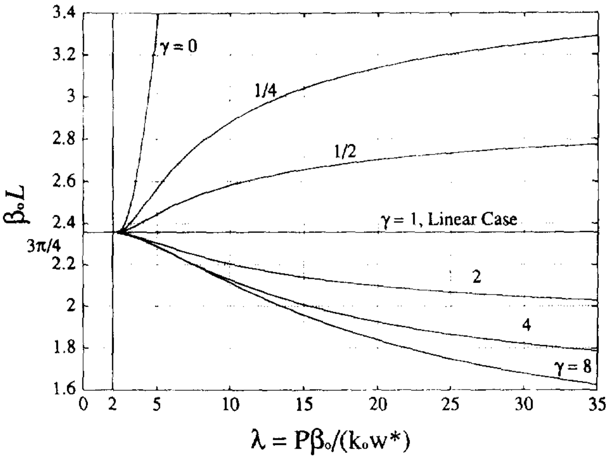

Fig. 10. Dependence of L on P and $\gamma = k_1/k_0$.

in Fig. 8 (elasto-plastic response). Since the presented solution is not valid for $\gamma = 0$, it was calculated for a very small value of γ. This case was also solved directly by replacing the first differential equation in (32) by

$$EIw_1^{iv} + p^* = 0 \quad 0 < x < a. \tag{32'}$$

The results agreed. The corresponding curve is shown in Fig. 10 as $\gamma = 0$.

Next, a similar analysis was conducted for bending moments, noting that $M = -EIw''$. The determined distances L_M of the first zero from the load P are shown in Fig. 11. Note the strong effect the base non-linearity, and the load P, have on the location of the first zero of the bending moment distribution, L_M.

Next, we establish the effect of the track base non-linearity on the rail bending moments. As an example, the largest bending moment which takes place at the load P will be determined.

Noting that $M(x) = -EIw''(x)$ and that $w(x)$ is given in eqn (34), it follows that

$$M_{max} = M(0) = -2EI\beta_1^2 A_4, \tag{39}$$

where A_4 is given in eqn (35). This moment expression is evaluated for a continuously welded 115 RE rail ($I = 2743$ cm^4) of a main-line track, with $E = 20{,}000$ kN cm^{-2}. For the

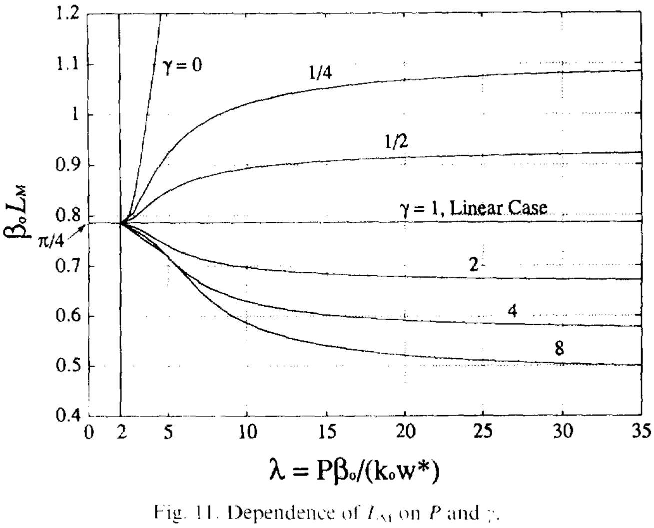

Fig. 11. Dependence of L_M on P and γ.

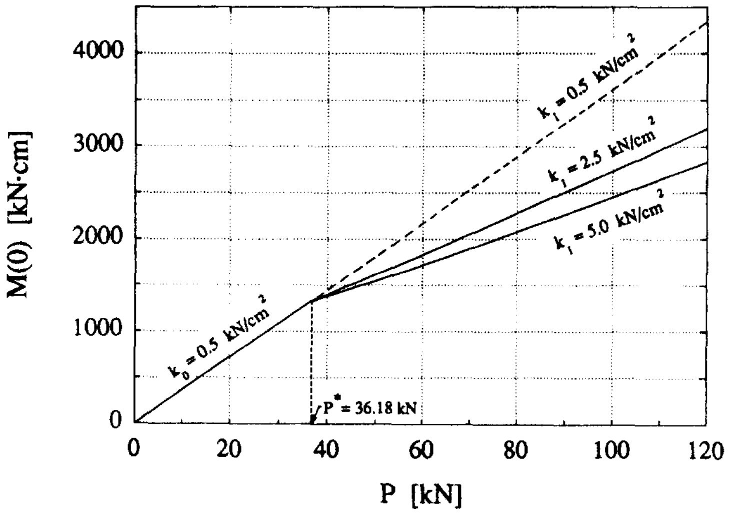

Fig. 12. Effect of stiffening base on the rail bending moment at P.

track base it is assumed that $k_0 = 0.5$ kN cm^{-2}, $k_1 = 2.5$ kN cm^{-2} or $k_1 = 5.0$ kN cm^{-2} and $w^* = 0.25$ cm, respectively. For a justification of these parameters refer to Kerr and Eberhardt (1992). The results of the numerical evaluation are shown in Fig. 12.

Note that for wheel loads $P > P^* = 2k_0w^*/\beta_0 = 36.18$ kN, the actual bending moments may be much smaller than those predicted by the linear analysis with $k_0 = 0.5$ kN cm^{-2}. For example, when $P = 100$ kN† and $k_1 = 2.5$ kN cm^{-2}, the largest bending moment $M(0)$ according to the linear analysis is 31.5% higher than the one obtained from the non-linear analysis. For $k_1 = 5.0$ kN cm^{-2} it is 47% higher. Since the bi-linear analysis represents closely the actual response in the field, it follows that the linear analysis with $k_0 = 0.5$ kN cm^{-2} substantially overestimates $M_{max} = M(0)$.

It may be shown (Kerr and Shenton, 1986; Kerr and Eberhardt, 1992), however, that for problems of this type the linear analysis grossly underestimates the contact pressure between beam and base at P.

The mechanical explanation of these analytical results is that, because of the stiffening of the base for $P > P^*$ in the vicinity of P, in an actual case the deflections and curvatures in this beam region are smaller and the contact pressures are larger than the ones that correspond to the linear analysis.

CONCLUSIONS

At first it was shown that the non-dependence of the separation points in the first problem (Fig. 1), and the location of the zero points in the second problem (Fig. 2), on the load intensity are a consequence of the linearity of the respective formulation.

To show the deficiencies that may result using a linear base response, a non-linearity was included in the Winkler foundation, by utilizing a bi-linear response. The following results were obtained. (1) For the finite beam that rests on the base, the load intensity affects the location of the point of separation of beam and base, and a "stiffening" of the base as well as an increasing load decreases the region of contact. (2) For the infinite beam that is attached to the base it was established that with increasing load and "stiffening" of the base, the distances of the zero locations get smaller, whereas for a "softening" base these distances increase, as compared to the linear case. (3) When the k value, which was determined from a test which utilized a relatively small load, is used in conjunction with the linear analysis, then for $P > P^*$ the bending moment $M_{max} = M(0)$ may be substantially overestimated. However, the contact pressure $P_{max} = p(0)$ may be grossly underestimated.

The above discussion suggests that when analysing continuously supported structures special attention must be devoted to the non-linear features of the base response.

† Note that currently the static wheel loads of freight cars in North America range between 140 and 174 kN.

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